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ON THE

# COLOR CORRECTION OF ACHROMATIC TELESCOPES.

BY WM. HARKNESS.

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*On the Color Correction of Achromatic Telescopes;*
by Wm. Harkness.

Although much has been written on the theory of the achromatic telescope, I am not aware that any attempt has hitherto been made to treat the color correction rigorously as a function of the wave length of the light; and, on that account, much obscurity, and some positive error, has crept into the text books on the subject.

The theory given in the following pages is based upon fundamental equations which neglect the thickness of the lenses, as has always been done heretofore, and which suffice to give the refractive indexes to scarcely more than four places of decimals. All the subsequent operations upon these equations are rigorously accurate; and it would have been useless to attempt greater precision in the refractive indexes, while the thickness of the lenses is neglected. As achromatic telescopic objectives are usually composed of two lenses, rarely of three, and hardly ever of a greater number; it has been thought sufficient to write the equations in the form applicable to triple objectives, but no difficulty will be experienced in extending them to a greater number of lenses, when necessary.

Our fundamental equations are

$$\frac{1}{f} = (\mu - 1)\left\{\frac{1}{r} + \frac{1}{\rho}\right\} \tag{1}$$

$$\mu = a + b\gamma^2 + c\gamma^4 \tag{2}$$

in which

$f$ = the principal focal distance of any lens.
$\mu$ = the refractive index of the lens.
$r$ = the radius of curvature of the first surface of the lens.
$\rho$ = the radius of curvature of the second surface of the lens.
$\lambda$ = the wave length of the light.
$\gamma = 1 \div \lambda$.
$a, b, c$ = certain coefficients, determined from not less than three properly situated values of $\mu$.

Equation (2) is Cauchy's dispersion formula. Now put

$$\frac{1}{r} + \frac{1}{\rho} = \mathrm{A} \tag{3}$$

and suppose a series of lenses, such that

$$\frac{1}{f_1} = (\mu_1 - 1)\mathrm{A}_1\,;\ \frac{1}{f_2} = (\mu_2 - 1)\mathrm{A}_2\,;\ \frac{1}{f_3} = (\mu_3 - 1)\mathrm{A}_3 \tag{4}$$

These lenses being very thin, let them be placed in contact with each other; and let the equivalent focal distance of the whole

combination be $f$. Then, by a well known optical theorem,

$$\frac{1}{f}=(\mu_1-1)A_1+(\mu_2-1)A_2+(\mu_3-1)A_3 \tag{5}$$

Substituting the values of $\mu_1$, $\mu_2$, $\mu_3$, from equation (2), putting

$$\left.\begin{aligned} C&=A_1(a_1-1)+A_2(a_2-1)+A_3(a_3-1) \\ D&=A_1b_1+A_2b_2+A_3b_3 \\ E&=A_1c_1+A_2c_2+A_3c_3 \end{aligned}\right\} \tag{6}$$

and arranging the terms according to the powers of $\gamma$, we have

$$f=\frac{1}{C+D\gamma^2+E\gamma^4} \tag{7}$$

This equation expresses the relation between the focal distance of the combination, and the wave length of the light. It shows that when white light enters an objective there will generally be an infinite number of foci, situated one behind the other, and all contained between the two values of $f$ which correspond to the limiting values of $\gamma$. For our purpose, however, it will be more convenient to consider $f$ as the ordinate, and $\gamma$ as the abscissa, of a curve which we will designate as the focal curve. To investigate its properties, we differentiate with respect to $f$ and $\gamma$, and obtain

$$\frac{df}{d\gamma}=-2\gamma f^2(D+2E\gamma^2) \tag{8}$$

Putting the left hand member of this expression equal to zero, we find

$$\gamma^2=-\frac{D}{2E} \tag{9}$$

Differentiating (8) a second time

$$\frac{d^2f}{d\gamma^2}=2f^3\gamma^2(2D+4E\gamma^2)^2-f^2(2D+12E\gamma^2) \tag{10}$$

Substituting the value of $\gamma^2$ from (9), this becomes

$$\frac{d^2f}{d\gamma^2}=4Df^2 \tag{11}$$

which shows that, so long as D remains positive, the curve is convex toward the objective, and the value of $\gamma$ given by equa- (9) corresponds to the minimum focal distance.

An achromatic objective, or more accurately, and with greater generality, a corrected objective, is one in which all rays of the kind for which the correction is made are brought to as nearly as possible the same focus. For example; if an objective is corrected for visual purposes, then the rays which produce the greatest effect upon the human eye must all be brought as nearly as possible to the same focus; or, if the objective is corrected for photographic purposes, then the rays which act

most energetically upon silver bromo-iodide must all be brought as nearly as possible to the same focus. This condition will evidently be fulfilled when the rays in question have the minimum focal distance; or in other words, when they satisfy equation (9). Thus it appears that this equation determines the correction of the objective, and for that reason it will be called the achromatic equation, and the particular value of $\gamma$ which satisfies it will be designated as $\gamma_0$.

To find the relative values of $A_1$, $A_2$, $A_3$, in a corrected objective, we substitute in (9) the values of D and E from (6). The resulting expression for the middle lens is

$$-A_2 = \frac{A_1(b_1 + 2c_1\gamma_0^2) + A_3(b_3 + 2c_3\gamma_0^2)}{(b_2 + 2c_2\gamma_0^2)} \qquad (12)$$

which shows that this lens must be of the opposite kind from the other two,—that is, if the first and third lenses are convex, the middle one must be concave; or *vice versa*.

To find the equivalent focal distance of the whole combination for the ray $\lambda_0$, (9) gives

$$D = -2E\gamma_0^2 \qquad (13)$$

Substituting this in (7)

$$f_0 = \frac{1}{C - E\gamma_0^4} \qquad (14)$$

Replacing C and E by their values from (6)

$$f_0 = \frac{1}{A_1(a_1 - c_1\gamma_0^4 - 1) + A_2(a_2 - c_2\gamma_0^4 - 1) + A_3(a_3 - c_3\gamma_0^4 - 1)} \qquad (15)$$

Substituting the value of $A_2$ from (12), and putting

$$\left.\begin{aligned} n &= \frac{A_3}{A_1} \\ L &= (a_1 - 1)(b_2 + 2c_2\gamma_0^2) - (a_2 - 1)(b_1 + 2c_1\gamma_0^2) + \gamma_0^4(b_1c_2 - b_2c_1) \\ M &= (a_3 - 1)(b_2 + 2c_2\gamma_0^2) - (a_2 - 1)(b_3 + 2c_3\gamma_0^2) + \gamma_0^4(b_3c_2 - b_2c_3) \end{aligned}\right\} \qquad (16)$$

we obtain finally

$$f_0 = \frac{b_2 + 2c_2\gamma_0^2}{A_1(L + nM)} \qquad (17)$$

The ordinate of the focal curve for the ray $\lambda_1$, is the difference between the focal lengths of the objective for the rays $\lambda_0$ and $\lambda_1$. To find it we have

$$\frac{1}{f_0} - \frac{1}{f_1} = (C + D\gamma_0^2 + E\gamma_0^4) - (C + D\gamma_1^2) + E\gamma_1^4) = D(\gamma_0^2 - \gamma_1^2) + (E\gamma_0^4 - \gamma_1^4) \qquad (18)$$

But

$$\frac{1}{f_0} - \frac{1}{f_1} = \frac{f_1 - f_0}{f_0 f_1} \qquad (19)$$

and putting $f_1 - f_0 = \Delta f_1$, this becomes

$$\Delta f_1 = f_0 f_1 \left\{ \frac{1}{f_0} - \frac{1}{f_1} \right\} \qquad (20)$$

Substituting for the quantity within the brackets, its value from (18); and replacing D and E by their equivalents from (6)

$$\Delta f_1 = f_0 f_1 \{(A_1 b_1 + A_2 b_2 + A_3 b_3)(\gamma_0^2 - \gamma_1^2) + (A_1 c_1 + A_2 c_2 + A_3 c_3)(\gamma_0^4 - \gamma_1^4)\} \quad (21)$$

Substituting the values of $A_2$ and $A_3 \div A_1$ from (12) and (16), and putting

$$\begin{aligned} N &= b_1 c_2 - b_2 c_1 \\ P &= b_3 c_2 - b_2 c_3 \end{aligned} \quad (22)$$

we obtain the important expression

$$\Delta f_1 = A_1 f_0 f_1 (\gamma_0^2 - \gamma_1^2)^2 \frac{N + nP}{b_2 + 2c_2\gamma_0^2} \quad (23)$$

If a star is viewed through a carefully focused achromatic telescope, and if the surface in the focus of the eye-piece is designated as the focal plane: then, of the infinite number of images which equation (7) shows will be formed, some will be situated before, and some behind the focal plane, but only one will coincide exactly with it. The cones of rays which form the images situated before and behind the focal plane will necessarily have a sensible diameter at their intersection with that plane, and their combined effect will be to produce a fringe of colored light around the image of the star, as seen through the eye-piece. This fringe is the secondary spectrum, and its magnitude, for light of any given wave length, will evidently depend upon the value of $\Delta f_1$. Hence, to destroy the secondary spectrum, $\Delta f_1$ must be made equal to zero. Equation (23) shows that this will be the case for a triple objective when

$$N + nP = 0 \quad (24)$$

or for a double objective when

$$N = 0 \quad (25)$$

As yet no materials have been discovered whose physical properties are such as to satisfy these conditions. We therefore proceed to investigate what form an objective constructed of any given materials must have in order to render the secondary spectrum a minimum.

Substituting in (23) the value of $A_1$ from (17), we find

$$\Delta f_1 = f_1 (\gamma_0^2 - \gamma_1^2)^2 \frac{(N + nP)}{(L + nM)} \quad (26)$$

In the right hand member of this equation, $n$ is the only quantity which depends upon the form of the objective. Considering it as variable, and differentiating, we obtain

$$\frac{d(\Delta f_1)}{dn} = f_1 (\gamma_0^2 - \gamma_1^2)^2 \frac{PL - MN}{(L + nM)^2} \quad (27)$$

To make $\Delta f_1$ a minimum, such a value must be attributed to $n$ as will reduce the right hand member of (27) to zero. This

condition gives at once, $n=\infty$; which will be the case when $r$ and $\rho$ are both infinite; as is evident from equations (16) and (3). The objective is then reduced to two lenses, and a piece of very thin plano-parallel glass. As the latter cannot appreciably affect the color correction, it may be dismissed from further consideration; and thus it appears that from any three pieces of glass suitable for making an objective, but not fulfilling the conditions necessary for the complete destruction of the secondary spectrum, it will always be possible to select two pieces from which a double objective can be made that will be superior to any triple objective made from all three of the pieces.

The focal curve being tangent to the focal plane at the point corresponding to the wave length $\lambda_0$; if we assume the spherical aberration to be perfectly corrected for light of all degrees of refrangibility; then the image of a star formed upon the focal plane by light of wave length $\lambda_0$ will be a point, and the linear semi-diameter of the image of the same star formed by light of wave length $\lambda_1$ will be the semi-diameter of the cone of rays of that wave length at the point where it cuts the focal plane. Therefore we have

$$f_1 : \alpha :: \Delta f_1 : s_0^{\text{ii}} \tag{28}$$

in which $\alpha$ is the semi-aperture of the objective, and $s_0^{\text{ii}}$ is the required semi-diameter of the cone of rays of wave length $\lambda_1$. Combining (28) with (26), we find

$$s_0^{\text{ii}} = \alpha(\gamma_0^2 - \gamma_1^2)^2 \frac{\text{N} + n\text{P}}{\text{L} + n\text{M}} \tag{29}$$

This is the expression for a triple objective. In the case of a double one, $n$ becomes zero, and (29) reduces to

$$s_0^{\text{ii}} = \alpha(\gamma_0^2 - \gamma_1^2)^2 \frac{\text{N}}{\text{L}} \tag{30}$$

which shows that in a double objective properly corrected for any given purpose, the linear semi-diameter of the secondary spectrum is absolutely independent, both of the focal length of the combination, and of the curves of its lenses; and depends solely upon the aperture of the combination, and the physical properties of the materials composing it.

If a telescope armed with an achromatic eye-piece is carefully focused upon a star, and then the image of the star is viewed through a prism held before the eye-piece; it will be seen that the eye does not adjust the focal plane tangent to the focal curve, but places it somewhat further from the objective, in such wise that the plane cuts the curve in two points, which we will designate as $\gamma_m$ and $\gamma_n$. For these points we must have

$$\frac{1}{\text{C} + \text{D}\gamma_m^2 + \text{E}\gamma_m^4} = \frac{1}{\text{C} + \text{D}\gamma_n^2 + \text{E}\gamma_n^4} \tag{31}$$

which gives

$$-\frac{D}{E} = \gamma_m^2 + \gamma_n^2 \tag{32}$$

But by (9) we have

$$\gamma_0^2 = -\frac{D}{2E} \tag{33}$$

Combining this with (32), we find

$$\gamma_0^2 = \tfrac{1}{2}(\gamma_m^2 + \gamma_n^2) \tag{34}$$

which gives the relation between $\gamma_0$ and any pair of points at which the focal plane may cut the focal curve.

We have next to consider how the value of $\gamma_m$ can be found; and for that purpose a method partly arithmetical, and partly graphical, seems most convenient. The data required are, the values of $\Delta f$ for a number of different values of $\gamma$, and the relative intensity of the light at each of these values of $\gamma$. The values of $\Delta f$ must be computed by means of equation (26); and the relative intensity of the light may either be determined experimentally, or taken from published tables. For visual intensity, the table given by Fraunhofer may be employed; and for photographic intensity, the curves published by Captain Abney contain all that is required. For the sake of definiteness, let us suppose that the value of $\gamma_m$ is to be determined for an objective corrected for visual purposes. We begin by laying down an axis of abscissas, and graduating it into a scale of wave lengths. Here, however, it must be observed that the brightness of any part of a spectrum depends not only upon the inherent brightness of the light at that point, but also upon the degree of dispersion employed. As Fraunhofer's determinations of the relative brightness of different parts of the spectrum were made with a flint glass prism having a refractive index of 1·63 for the ray D; and as such an instrument produces much greater dispersion at the violet end of the spectrum than at the red end; it follows that our scale of wave lengths must be, not a scale of equal parts, but such a scale as existed in the spectrum employed by Fraunhofer. The wave length of the brightest ray is approximately 5688, and through that point in the scale, and at right angles to the axis of abscissas, the axis of ordinates must be drawn. Then, from the computed values of $\Delta f$, a sufficient number of points must be laid down to determine the focal curve, and that curve must be drawn. At the points whose wave lengths correspond to the principal Fraunhofer lines, lines must be drawn through the focal curve, parallel to the axis of ordinates; the length of each line being proportional to the relative brightness of the spectrum at the point where it is situated, and the center of each line coinciding accurately with the focal curve. Through

the extremities of these lines a closed curve must be drawn. The figure thus obtained will be termed the illumination diagram, because it exhibits the amount and distribution of the light at the focus of the objective. The eye will necessarily place the focal plane in the position where this light will produce the greatest effect upon the retina; which is equivalent to saying that the focal plane must pass through the center of gravity of the diagram. Hence, to find the position of the focal plane, we have only to cut out the diagram (which should be drawn upon rather stiff paper), and balance it upon a knife edge held parallel to the axis of abscissas. The reciprocals of the wave lengths of the points of intersection of the knife edge with the focal curve will then be the values of $\gamma_m$ and $\gamma_n$.

The method just explained may be employed to determine the difference between the positions of the principal focus of the same telescope when used for different purposes. For example, if it were required to find the interval between the visual and photographic foci of a telescope, two illumination diagrams would be drawn—one for the visual, and the other for the photographic rays—and the difference between the positions of the focal plane in the two diagrams would be the required difference of foci.

As the magnitude of the secondary spectrum of a star is measured by the semi-diameter (at the point where it intersects the focal plane) of the cone of rays having the maximum focal distance; it follows that in an objective corrected for visual purposes, the secondary spectrum is diminished by the fact that the eye places the focal plane somewhat further from the objective than the apex of the focal curve. To find the amount of this diminution, we remark that for light of wave lengths corresponding to the points where the focal plane cuts the focal curve, the semi-diameter of the cone of rays is zero; while for light of any other wave length, the semi-diameter of the cone of rays, at the point where it intersects the focal plane, is proportional to the distance between that plane and the point of the focal curve corresponding to the wave length of the light. Hence, the effect of moving the focal plane into a position further from the objective than the apex of the focal curve, will be to diminish $s_0^{\text{ii}}$ by a constant which is numerically equal to the value of $s_0^{\text{ii}}$ for light whose wave length is that of the point at which the focal plane intersects the focal curve. Modifying equation (30) in accordance with these principles, it becomes

$$s_m^{\text{ii}} = \alpha\{(\gamma_0^2 - \gamma_1^2)^2 - (\gamma_0^2 - \gamma_m^2)^2\} \frac{\text{N}}{\text{L}} \tag{35}$$

in which $\gamma_m$ is the reciprocal of the wave length corresponding to either of the two points in which the focal plane cuts the

focal curve; and $s_m^{\text{II}}$ is the semi-diameter, at the point where it cuts the focal plane, of the cone of rays whose wave length is $\lambda_1$.

The exact nature of the color correction of a telescope can be determined by placing the focal plane in a number of different positions, and observing the corresponding values of $\gamma_m$ and $\gamma_n$. These values being substituted in equation (34), several independent values of $\gamma_0$ can be deduced, the mean of which will probably be very near the truth.

The conclusions reached in the preceding pages may be summed up as follows:

1st. From any three pieces of glass suitable for making a corrected objective, but not fulfilling the conditions necessary for the complete destruction of the secondary spectrum, it will always be possible to select two pieces from which a double objective can be made that will be superior to any triple objective made from all three of the pieces.

2d. The color correction of an objective is completely defined by stating the wave length of the light for which it gives the minimum focal distance.

3d. An objective is properly corrected for any given purpose when its minimum focal distance corresponds to rays of the wave length which is most efficient for that purpose. For example, in an objective corrected for visual purposes the rays which seem brightest to the human eye should have the minimum focal distance; while in an objective intended for photographic purposes the rays which act most intensely upon silver bromo-iodide should have the minimum focal distance.

4th. In double achromatic objectives the secondary spectrum (or in other words, the diameter, at its intersection with the focal plane, of the cone of rays having the maximum focal distance), is absolutely independent both of the focal length of the combination, and of the curves of its lenses; and depends solely upon the aperture of the combination, and the physical properties of the materials composing it.

5th. When the focal curve of an objective is known; and the relative intensity, for the purpose for which the objective is corrected, of light of every wave length is also known; then the exact position which the focal plane should occupy can readily be calculated.

6th. It may be remarked incidentally that in an objective corrected for photographic purposes, the interval between the maximum and minimum focal distances is less than in one corrected for visual purposes. Hence, a photographic objective has less secondary spectrum, and is better adapted to spectroscopic work, than a visual objective.

Washington, May 24. 1879.